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Connections between uniformity and aberration in general multi-level factorials

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Abstract Discrepancy is a kind of important measure used in experimental designs. Among various existing discrepancies, the discrete discrepancy, centered L_2- (CD_2) and wrap-around L_2-discrepancy (WD_2) have been well justified and widely used. In this paper, using the second-order polynomials of indicator functions for these three discrepancies, we investigate the close relationships between them and the generalized wordlength pattern, and provide some conditions under which a design having one of these minimum discrepancies is equivalent to having generalized minimum aberration (GMA). These results provide further justifications for the criterion of GMA in terms of uniformity. In addition, the expressions of the discrepancies in the quadratic forms of the indicator functions are useful for us to find optimal designs under any of them.

Keywords Discrepancy · Generalized minimum aberration · Generalized wordlength pattern · Indicator function · Uniformity

1 Introduction

Fractional factorial designs have become important in all kinds of investigations and have been found to be of great utility in many areas of experimentation. A fundamental and practical question for fractional factorial designs is how to choose a good design from a set of candidates. There are many useful criteria defined from different viewpoints for comparing fractional factorial designs. The maximum resolution criterion proposed by Box and Hunter (1961) and the minimum aberration criterion

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proposed by Fries and Hunter (1980) have become the standard criteria for optimal factor assignment. Recently, the minimum aberration criterion was generalized to the nonregular case by Tang and Deng (1999), Ma and Fang (2001), Xu and Wu (2001) and Cheng and Ye (2004), i.e., the generalized minimum aberration (GMA) criterion which sequentially minimizes the generalized wordlength pattern. The discrepancy is another important measure used for evaluating factorial designs (Hickernell 1998; Liu 2002; Hickernell and Liu 2002; Fang et al. 2006). It measures how much the empirical distribution of the design points departs from the uniform distribution (Hickernell 1999). Three most important discrepancies are the centered L_2-discrepancy (CD_2), wrap-around L_2-discrepancy (WD_2) and discrete discrepancy. Fang and Mukerjee (2000), Hickernell and Liu (2002), Liu (2002), Liu and Hickernell (2002), Fang et al. (2003), Ma et al. (2003), Qin and Fang (2004), Liu et al. (2005, 2006), Qin and Li (2006), Qin and Ai (2007), Qin et al. (2009) explored the relationships among these discrepancies and some orthogonality measures, including the generalized wordlength pattern.

This paper is organized as follows. In Sect. 2, we introduce some notations and the definitions of indicator function, GMA, and those discrepancies. The link and equivalency for some cases between the discrete discrepancy and aberration are established in Sect. 3. Section 4 explores the connections between CD_2/WD_2 and aberration, and presents some cases when they are equivalent to each other. There are some concluding remarks in the last section.

2 Preliminaries

Let n be the number of runs and s be the number of factors of a design, where each factor has q levels. Denote the set of all the $n \times s$ design matrices by $\mathcal{D}(n; q^s)$. A design $A \in \mathcal{D}(n; q^s)$ is called an orthogonal design and denote by $L_n(q^s)$ if it has the property that in any two columns every possible 2-tuple occurs an equal number of times. We denote the full factorial design of s q-level factors by X, with the level combinations arranged in the lexicographic order, i.e., the first level combination is $(0, \dots, 0, 0)$, the second one is $(0, \dots, 0, 1)$, the qth one is $(0, \dots, 0, q-1)$ and the last one is $(q-1, \dots, q-1, q-1)$. Let $\mathbf{1}_q$ be the $q \times 1$ vector with all elements unity, I_q be the identity matrix of order q. For any run $\boldsymbol{x}$ of design A, we know $\boldsymbol{x}$ is from X, but a design point $\boldsymbol{x}$ in X may not appear or appear more than once in A. The number of appearances of $\boldsymbol{x}$ in A, denoted by $f_A(\boldsymbol{x})$, is called the indicator function of design A.

The description of fractional factorial designs using the polynomial representation of their indicator functions has been introduced for two-level factorial designs by Fontana et al. (2000) and Ye (2003), and generalized to mixed-level factorial designs by Cheng and Ye (2004) and Pistone and Rogantin (2008). For each factor X_i of X, $i = 1, \dots, s$, define a set of orthogonal contrasts $C_0^i(x),\ C_1^i(x), \dots, C_{q-1}^i(x)$ such that

$$\sum_{x=0}^{q-1} C_u^i(x) C_v^i(x) = \begin{cases} 0, & \text{if } u \neq v, \\ q, & \text{if } u = v. \end{cases}$$

In statistical analysis, $C_0^i(x) = 1$ is often adopted to represent a constant term, $x = 0, \ldots, q-1$. An orthogonal contrast basis on X is defined as $p_{\boldsymbol{l}}(\boldsymbol{x}) = \prod_{i=1}^s C_{l_i}^i(x_i)$ for $\boldsymbol{l} \in L = \{\boldsymbol{l} = (l_1, \ldots, l_s) : l_i = 0, \ldots, q-1\}$ and $\boldsymbol{x} \in X$. Obviously, for $\boldsymbol{l}, \boldsymbol{m} \in L$

$$\sum_{\boldsymbol{x} \in X} p_{\boldsymbol{l}}(\boldsymbol{x}) p_{\boldsymbol{m}}(\boldsymbol{x}) = \begin{cases} 0, & \text{if } \boldsymbol{l} \neq \boldsymbol{m}, \\ q^s, & \text{if } \boldsymbol{l} = \boldsymbol{m}. \end{cases}$$

Therefore, the indicator function $f_A(\boldsymbol{x})$ on X can be represented by a unique linear combination of these contrasts. Since each contrast is a monomial, $f_A(\boldsymbol{x})$ has the polynomial form

$$f_A(\boldsymbol{x}) = \sum_{\boldsymbol{l} \in L} b_{\boldsymbol{l}} p_{\boldsymbol{l}}(\boldsymbol{x}),$$

which is called the indicator polynomial function of A and the coefficients $\{b_{\boldsymbol{l}}, \boldsymbol{l} \in L\}$ are uniquely determined by

$$b_{\boldsymbol{l}} = \frac{1}{q^s} \sum_{\boldsymbol{x} \in A} p_{\boldsymbol{l}}(\boldsymbol{x}). \tag{1}$$

Let F_A be an $n \times q^s$ matrix, with rows indexed by $i, i = 1, \ldots, n$, and columns indexed by $\boldsymbol{j} = (j_1, \ldots, j_s)$ in the lexicographic order of $j_t = 0, \ldots, q-1$, $t = 1, \ldots, s$. If the ith run of A takes level combination $(j_1, \ldots, j_s)$, then $F_{i,(j_1,\ldots,j_s)} = 1$, and 0 otherwise. We call F_A the indicator matrix of design A. Let $P = (p_{\boldsymbol{l}})_{\boldsymbol{l} \in L}$ be the contrast matrix of the full factorial design X, where $p_{\boldsymbol{l}} = (p_{\boldsymbol{l}}(\boldsymbol{x}_1), \ldots, p_{\boldsymbol{l}}(\boldsymbol{x}_{q^s}))'$ and $\boldsymbol{x}_i$ is the ith run of X, $i = 1, \ldots, q^s$, then the contrast matrix of A can be represented by $F_A P$. From (1), we know that $b_{\boldsymbol{l}} = 0$ implies that the column contrast vector $F_A p_{\boldsymbol{l}}$ has zero correlation with the grand mean term $\mathbf{1}_n$ for design A. Various statistical properties of designs can be studied through the $b_{\boldsymbol{l}}$'s. The GMA criterion due to Xu and Wu (2001) has been redefined by Cheng and Ye (2004) using the $b_{\boldsymbol{l}}$'s. It is to sequentially minimize the generalized wordlength pattern $(\alpha_0(A), \ldots, \alpha_s(A))$, where

$$\alpha_r(A) = \sum_{\|\boldsymbol{l}\|_0 = r} \left(\frac{b_{\boldsymbol{l}}}{b_{\mathbf{0}}}\right)^2, \quad r = 0, \ldots, s, \tag{2}$$

$\|\boldsymbol{l}\|_0$ denotes the number of nonzero elements in $\boldsymbol{l}$. Xu and Wu (2001) pointed that $\alpha_r(A)$ is independent of the choice of orthogonal contrasts following Remark 2.3.1 of Dey and Mukerjee (1999).

The discrepancy is another important measure used for evaluating designs. Fang and Qin (2003) showed that the CD_2 and WD_2 discrepancies can be expressed as the second-order polynomials of the indicator functions, respectively.

Lemma 1 (Fang and Qin 2003) *Let* $\boldsymbol{f}_A = (f_A(\boldsymbol{x}_1), \ldots, f_A(\boldsymbol{x}_{q^s}))'$ *with* $\boldsymbol{x}_i$ *being the* i*th run of* X *for* $i = 1, \ldots, q^s$, *then for any* $A \in \mathcal{D}(n; q^s)$, *we have*

$$[WD_2(A)]^2 = -\left(\frac{4}{3}\right)^s + \frac{1}{n^2}\boldsymbol{f}'_A W_s \boldsymbol{f}_A, \tag{3}$$

where $W_s = \otimes^s W_0$, $W_0 = (w_{ij})$ *and* $w_{ij} = 3/2 - |i-j|(q-|i-j|)/q^2$, $i, j = 0, \ldots, q-1$; *and*

$$[CD_2(A)]^2 = \left(\frac{13}{12}\right)^s - \frac{2}{n}\boldsymbol{e}'_s \boldsymbol{f}_A + \frac{1}{n^2}\boldsymbol{f}'_A C_s \boldsymbol{f}_A, \tag{4}$$

where $\boldsymbol{e}_s = \otimes^s \boldsymbol{e}_0$, $C_s = \otimes^s C_0$, $\boldsymbol{e}_0 = (e_0, \ldots, e_{q-1})'$, $C_0 = (c_{ij})$,

$$e_i = 1 + \frac{1}{2}\left|\frac{2i+1-q}{2q}\right| - \frac{1}{2}\left|\frac{2i+1-q}{2q}\right|^2, \text{ and}$$
$$c_{ij} = 1 + \frac{1}{2}\left|\frac{2i+1-q}{2q}\right| + \frac{1}{2}\left|\frac{2j+1-q}{2q}\right| - \frac{1}{2}\left|\frac{i-j}{q}\right|, \quad i, j = 0, \ldots, q-1.$$

The discrete discrepancy has received a great deal of attention in recent literature in experimental designs (Hickernell and Liu 2002; Liu 2002; Liu and Hickernell 2002, 2006; Fang et al. 2002, 2003; Qin and Fang 2004; Liu et al. 2005). It can be utilized as a uniformity measure for comparing and evaluating factorial designs.

3 Connection between discrete discrepancy and aberration

For a given design $A \in \mathcal{D}(n; q^s)$, its discrete discrepancy value, denote by $D(A; a, b)$, can be calculated as follows

$$[D(A; a, b)]^2 = -\left[\frac{a + (q-1)b}{q}\right]^s + \frac{b^s}{n^2}\sum_{k=1}^n \sum_{l=1}^n \left(\frac{a}{b}\right)^{\lambda_{kl}}, \tag{5}$$

where a and b are two constants with $a > b > 0$, λ_{kl} is the number of coincidences between the kth and lth runs of A (Fang et al. 2002, 2003). In fact, from

$$\sum_{k=1}^n \sum_{l=1}^n \left(\frac{a}{b}\right)^{\lambda_{kl}} = \sum_{k=1}^n \sum_{l=1}^n \prod_{j=1}^s \left(\frac{a}{b}\right)^{\delta_{a_{kj}a_{lj}}},$$

where δ_{xy} denotes the Kronecker delta function and $A = (a_{ij})$, we can easily obtain that

Lemma 2 *For any* $A \in \mathcal{D}(n; q^s)$, *we have*

$$[D(A; a, b)]^2 = -\left[\frac{a + (q-1)b}{q}\right]^s + \frac{b^s}{n^2}\boldsymbol{f}'_A D_s \boldsymbol{f}_A, \tag{6}$$

where $D_s = \otimes^s D_0$, $D_0 = (d_{ij})$ *and* $d_{ij} = a/b$, *if* $i = j$ *and* 1 *otherwise, for* $i, j = 0, \ldots, q-1$.

Note that each of the expressions (3), (4) and (6) contains a quadratic form of the indicator functions, so without loss of generality, we now discuss the properties of function

$$y = \boldsymbol{f}'_A B \boldsymbol{f}_A, \tag{7}$$

where B is a given matrix. First we have

Theorem 1 *If* B_0 *is a* $q \times q$ *positive definite matrix with an eigenvector* $\mathbf{1}_q$*, the corresponding eigenvalue is* λ_0*, and the other eigenvalues are* $\lambda_1, \ldots, \lambda_{q-1}$*, then*

$$B_0 = \sum_{i=0}^{q-1} \lambda_i T_i T_i', \tag{8}$$

and the contrast matrix of the full factorial design X *can be taken as* $P = (p_{\boldsymbol{l}})_{\boldsymbol{l} \in L} = \otimes^s \sqrt{q} T$*, where* $T = (T_0, \ldots, T_{q-1})$, $T_0 = 1/\sqrt{q}\mathbf{1}_q$ *and* T_i *is the unit eigenvector of* B_0 *corresponding to* λ_i, $i = 1, \ldots, q-1$*. Further, if* $B_s = \otimes^s B_0$*, then* $y = \boldsymbol{f}'_A B_s \boldsymbol{f}_A$ *can be expressed as*

$$y = \sum_{\boldsymbol{l} \in L} n^2 \lambda^2(\boldsymbol{l}) \left(\frac{b_{\boldsymbol{l}}}{b_{\boldsymbol{0}}} \right)^2, \tag{9}$$

where $\lambda^2(\boldsymbol{l}) = \prod_{i=1}^s (\lambda_{l_i}/q)$.

Proof Equation (8) is obvious. We only need to show that (9) is true. In fact,

$$\begin{aligned} y &= \boldsymbol{f}'_A B_s \boldsymbol{f}_A = \boldsymbol{f}'_A \left[\bigotimes^s \left(\sum_{i=0}^{q-1} \lambda_i T_i T_i' \right) \right] \boldsymbol{f}_A \\ &= \boldsymbol{f}'_A \left[\sum_{\boldsymbol{l} \in L} \left(\bigotimes_{i=1}^s \lambda_{l_i} T_{l_i} T_{l_i}' \right) \right] \boldsymbol{f}_A = \boldsymbol{f}'_A \sum_{\boldsymbol{l} \in L} \left[\left(\bigotimes_{i=1}^s \sqrt{\lambda_{l_i}} T_{l_i} \right) \left(\bigotimes_{i=1}^s \sqrt{\lambda_{l_i}} T_{l_i} \right)' \right] \boldsymbol{f}_A \\ &= \boldsymbol{f}'_A \sum_{\boldsymbol{l} \in L} \left[\left(\prod_{i=1}^s \sqrt{\frac{\lambda_{l_i}}{q}} \right) p_{\boldsymbol{l}} \right] \left[\left(\prod_{i=1}^s \sqrt{\frac{\lambda_{l_i}}{q}} \right) p_{\boldsymbol{l}} \right]' \boldsymbol{f}_A \\ &= \sum_{\boldsymbol{l} \in L} \left[\left(\prod_{i=1}^s \sqrt{\frac{\lambda_{l_i}}{q}} \right) \boldsymbol{f}'_A p_{\boldsymbol{l}} \right]^2 = \sum_{\boldsymbol{l} \in L} \left[\left(\prod_{i=1}^s \sqrt{\frac{\lambda_{l_i}}{q}} \right) (q^s b_{\boldsymbol{l}}) \right]^2 \\ &= \sum_{\boldsymbol{l} \in L} \lambda^2(\boldsymbol{l}) (q^s b_{\boldsymbol{l}})^2 = \sum_{\boldsymbol{l} \in L} n^2 \lambda^2(\boldsymbol{l}) \left(\frac{b_{\boldsymbol{l}}}{b_{\boldsymbol{0}}} \right)^2, \end{aligned}$$

where the last equality follows from the fact that $b_{\boldsymbol{0}} = n/q^s$. Thus, we complete the proof.

Further, from the two expressions in (2) and (9), we have

Corollary 1 *If in Theorem 1,* $\lambda_1 = \cdots = \lambda_{q-1}$*, then*

$$y = \boldsymbol{f}'_A B_s \boldsymbol{f}_A = n^2 \left(\frac{\lambda_0}{q}\right)^s \sum_{r=0}^{s} \left(\frac{\lambda_1}{\lambda_0}\right)^r \alpha_r(A), \tag{10}$$

where $\alpha_r(A)$ *is defined in* (2).

Now, from Corollary 1 and Lemma 2, we can obtain the following connection between the discrete discrepancy and generalized wordlength pattern.

Theorem 2 *For any* $A \in \mathcal{D}(n; q^s)$*, we have*

$$[D(A; a, b)]^2 = -\left[\frac{a+(q-1)b}{q}\right]^s + \left[\frac{a+(q-1)b}{q}\right]^s \sum_{r=0}^{s} \left[\frac{a-b}{a+(q-1)b}\right]^r \alpha_r(A), \tag{11}$$

$$= \left[\frac{a+(q-1)b}{q}\right]^s \sum_{r=1}^{s} \left[\frac{a-b}{a+(q-1)b}\right]^r \alpha_r(A). \tag{12}$$

Proof Since $a > b > 0$, D_0 is a positive definite matrix. From

$$D_0 \mathbf{1}_q = \left(\frac{a}{b} + q - 1\right) \mathbf{1}_q,$$

we know that $\mathbf{1}_q$ is an eigenvector of D_0, and the corresponding eigenvalue is $a/b + q - 1$. In fact, D_0 has two different eigenvalues, i.e., $\lambda_0 = a/b + q - 1,\ \lambda_1 = \cdots = \lambda_{q-1} = a/b - 1$. Based on Corollary 1 and Lemma 2, the result follows directly.

Note that (12) was also obtained by Qin and Fang (2004), here we present this alternative proof as it appears a bit more intuitive.

From Theorem 2, we know that the coefficient of $\alpha_r(A)$ in $[D(A; a, b)]^2$ decreases exponentially with r, so the design with less aberration tends to have smaller $[D(A; a, b)]^2$. This shows that uniform designs under $[D(A; a, b)]^2$ and GMA designs are strongly related to each other. But the $[D(A; a, b)]^2$ criterion does not completely agree with GMA, i.e., there may exist two designs A_1 and A_2 satisfying $[D(A_1; a, b)]^2 > [D(A_2; a, b)]^2$, but A_1 has less aberration than A_2. When are those two criteria equivalent to each other? The next theorem provides a condition, the proof is given in the Appendix.

Theorem 3 *Suppose* A_1 *and* A_2 *are two designs from* $\mathcal{D}(n; q^s)$*, both of which have no replicates. If*

$$\frac{a+(q-1)b}{a-b} - 1 = \frac{qb}{a-b} \geq n(q^s - n),$$

then $[D(A_1; a, b)]^2 < [D(A_2; a, b)]^2$ *is equivalent to* A_1 *having less aberration than* A_2.

Fang et al. (2000) conjectured an "equivalence theorem" between the uniformity of experimental points over the domain and the design orthogonality. They showed numerically that uniformity of experimental points over the domain can imply design orthogonality and conjectured that every orthogonal design can be obtained by minimizing some measure of uniformity. Ma et al. (2003) showed that the conjecture is only true in some special cases under CD_2. From Theorem 3, we know that if orthogonal designs $L_n(q^s)$ without replicates exist and $qb/(a-b) \geq n(q^s - n)$, then the uniform design under $[D(A; a, b)]^2$ is an orthogonal design and has GMA among all designs in $\mathcal{D}(n; q^s)$ without replicates.

From Lemma 3 in the Appendix, we can obtain the following result.

Theorem 4 (i) *Suppose* $A_1, A_2 \in \mathcal{D}(n; q^s)$ *and* $k\alpha_r(A_i)$ *are all integers for* $r = 1, \ldots, s$, *and* $i = 1, 2$, *where* k *is some constant. If* $qb/(a-b) \geq \max\{k\alpha_r(A_i), r = 1, \ldots, s, i = 1, 2\}$, *then* $[D(A_1; a, b)]^2 < [D(A_2; a, b)]^2$ *is equivalent to* A_1 *having less aberration than* A_2.

(ii) *Suppose* A_1 *and* A_2 *are two regular designs from* $\mathcal{D}(n; q^s)$. *If* $qb/(a-b) \geq \max\{\alpha_r(A_i), r = 1, \ldots, s, i = 1, 2\}$, *then* $[D(A_1; a, b)]^2 < [D(A_2; a, b)]^2$ *is equivalent to* A_1 *having less aberration than* A_2.

(iii) *Suppose* $A_1, A_2 \in \mathcal{D}(n; q^s)$. *If* $qb/(a-b) \geq \max\{\alpha_r(A_i), r = 1, \ldots, s, i = 1, 2\}$ *and there exists a positive integer* t *such that* $\alpha_r(A_1) = \alpha_r(A_2)$ *for* $r < t$ *and* $\alpha_t(A_1) \leq \alpha_t(A_2) - 1$, *then* $[D(A_1; a, b)]^2 < [D(A_2; a, b)]^2$.

The CD_2 and WD_2 discrepancies also have similar properties which will be investigated in the following section.

4 Connections between CD_2/WD_2 and aberration

Now, we consider the other two important discrepancies, the CD_2 and WD_2. From (3), (4), Theorem 1 and Corollary 1, we can obtain the following corollaries.

Corollary 2 *For any* $A \in \mathcal{D}(n; q^s)$,

$$[WD_2(A)]^2 = -\left(\frac{4}{3}\right)^s + \sum_{\boldsymbol{l} \in L} \lambda^2(\boldsymbol{l}) \left(\frac{b_{\boldsymbol{l}}}{b_{\boldsymbol{0}}}\right)^2, \tag{13}$$

where $\lambda^2(\boldsymbol{l}) = \prod_{i=1}^s [\lambda_{l_i}(W_0)/q]$ *and* $\lambda_i(W_0)$ *denotes the* i*th eigenvalue of* W_0.

Corollary 3 (i) *For* $A \in \mathcal{D}(n; 2^s)$,

$$[WD_2(A)]^2 = -\left(\frac{4}{3}\right)^s + \left(\frac{11}{8}\right)^s \sum_{r=0}^{s} \frac{\alpha_r(A)}{11^r}, \quad and \tag{14}$$

$$[CD_2(A)]^2 = \left(\frac{13}{12}\right)^s - 2\left(\frac{35}{32}\right)^s + \left(\frac{9}{8}\right)^s \sum_{r=0}^{s} \frac{\alpha_r(A)}{9^r}. \tag{15}$$

(ii) *For* $A \in \mathcal{D}(n; 3^s)$,

$$[WD_2(A)]^2 = -\left(\frac{4}{3}\right)^s + \left(\frac{73}{54}\right)^s \sum_{r=0}^{s} \left(\frac{4}{73}\right)^r \alpha_r(A). \tag{16}$$

Note that (15) was also obtained by Fang and Mukerjee (2000) for the regular 2-level case, and by Ye (2003) for the general 2-level case. The results in Corollary 3 were also provided by Ma and Fang (2001) for general 2- and 3-level cases, but they only proved (14) for the regular case, not for the general case.

From Corollary 3, we notice that the coefficient of $\alpha_r(A)$ in $[WD_2(A)]^2$ or $[CD_2(A)]^2$ decreases exponentially with r, so the design with less aberration tends to have smaller $[WD_2(A)]^2$ or $[CD_2(A)]^2$. Similar to the discrete discrepancy, uniform designs under $[WD_2(A)]^2$ or $[CD_2(A)]^2$ and GMA designs are also strongly related to each other. In fact, we have the following theorem, which shows when $[WD_2(A)]^2$ or $[CD_2(A)]^2$ agrees with GMA.

Theorem 5 (i) *Suppose* $A_1, A_2 \in \mathcal{D}(n; q^s)$ *and* $k\alpha_r(A_i)$ *are all integers for* $r = 1, \ldots, s$, *and* $i = 1, 2$, *where k is some constant, then*

(1) *if* $q = 2$ *and* $8 \geq \max\{k\alpha_r(A_i), r = 1, \ldots, s, i = 1, 2\}$, *then* $[CD_2(A_1)]^2 < [CD_2(A_2)]^2$ *is equivalent to* A_1 *having less aberration than* A_2;

(2) *if* $q = 2$ *and* $10 \geq \max\{k\alpha_r(A_i), r = 1, \ldots, s, i = 1, 2\}$, *then* $[WD_2(A_1)]^2 < [WD_2(A_2)]^2$ *is equivalent to* A_1 *having less aberration than* A_2;

(3) *if* $q = 3$ *and* $69/4 \geq \max\{k\alpha_r(A_i), r = 1, \ldots, s, i = 1, 2\}$, *the* $[WD_2(A_1)]^2 < [WD_2(A_2)]^2$ *is equivalent to* A_1 *having less aberration than* A_2.

(ii) *Suppose* $A_1, A_2 \in \mathcal{D}(n; q^s)$ *and there exists a positive integer t such that* $\alpha_r(A_1) = \alpha_r(A_2)$ *for* $r < t$ *and* $\alpha_t(A_1) \leq \alpha_t(A_2) - 1$, *then*

(1) *if* $q = 2$ *and* $8 \geq \max\{\alpha_r(A_i), r = 1, \ldots, s, i = 1, 2\}$, *then* $[CD_2(A_1)]^2 < [CD_2(A_2)]^2$;

(2) *if* $q = 2$ *and* $10 \geq \max\{\alpha_r(A_i), r = 1, \ldots, s, i = 1, 2\}$, *then* $[WD_2(A_1)]^2 < [WD_2(A_2)]^2$;

(3) *if* $q = 3$ *and* $69/4 \geq \max\{\alpha_r(A_i), r = 1, \ldots, s, i = 1, 2\}$, *then* $[WD_2(A_1)]^2 < [WD_2(A_2)]^2$.

The proof of Theorem 5 follows from Lemma 3 in the Appendix and is similar to that of Theorem 3, thus we omit it.

Note that, from Theorem 2 and Corollary 3, the equivalencies between the discrete discrepancy and CD_2 as well as WD_2 for some specific values of a, b and q will be easily derived, just as Qin and Fang (2004) showed in their Theorem 5.

5 Concluding remarks

This paper expresses the discrete discrepancy, CD_2 and WD_2 in the second-order polynomials of the indicator functions, and then discusses the relationships between these discrepancies and the generalized wordlength pattern. It also presents some cases when these uniformity criteria are equivalent to the aberration criterion, respectively. The close relationships among these criteria further show that the uniformity criteria can be utilized to compare fractional factorial designs and provide an additional rationale for employing uniform designs. All these results show that orthogonality is strongly related to uniformity, and provide some further justifications for the criterion of GMA in terms of uniformity.

In addition, the expressions of the discrete discrepancy and WD_2 in the quadratic forms of the indicator functions are useful for us to find optimal designs under each of the criteria. In fact, in order to find a design minimizing (7), we need only solve

$$\begin{cases} \min\limits_{\boldsymbol{f}_A} \boldsymbol{f}'_A B \boldsymbol{f}_A, \\ \text{s.t.} \ \ \boldsymbol{f}'_A \mathbf{1} = n, \ \boldsymbol{f}_A(\boldsymbol{x}_i) = 0, \ldots, n. \end{cases} \tag{17}$$

One such approach is provided in Sun et al. (2009), and many optimal designs under GMA as well as a uniformity criterion are tabulated there.

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Appendix

For proving Theorem 3, we need the following lemma.

Lemma 3 *Suppose a_i and b_i are all nonnegative numbers and $a_i, b_i \le m-1$, for $i = 0, \ldots, k$.*

(i) *If $a_k \le b_k - 1$, then $\sum_{i=0}^k a_i m^i < \sum_{i=0}^k b_i m^i$.*

(ii) *If a_i and b_i are integers with $a_k \ne b_k$, then $\sum_{i=0}^k a_i m^i < \sum_{i=0}^k b_i m^i$ if and only if $a_k < b_k$.*

Proof (i) can be proved from

$$\begin{aligned} \sum_{i=0}^{k} a_i m^i &= a_k m^k + \sum_{i=0}^{k-1} a_i m^i \le a_k m^k + (m-1)\sum_{i=0}^{k-1} m^i \\ &= a_k m^k + m^k - 1 < b_k m^k \le \sum_{i=0}^{k} b_i m^i. \end{aligned}$$

And (ii) follows from (i) directly. Thus the conclusion is true.

Now, let us prove Theorem 3.

Proof of Theorem 3 From the proof of Theorem 2 in Hickernell and Liu (2002), we have

$$\alpha_r(A) = \frac{1}{n^2} \sum_{|\boldsymbol{u}|=r} \sum_{i,k=1}^{n} \prod_{l \in \boldsymbol{u}} (-1 + q\delta_{a_{il}a_{kl}}), \tag{18}$$

where $\boldsymbol{u}$ is a subset of $\{1, \ldots, s\}$, $|\boldsymbol{u}|$ denotes the cardinality of $\boldsymbol{u}$, and $A = (a_{ij})$, thus $n^2\alpha_r(A)$ is an integer. In addition, from Theorem 4.1 of Cheng and Ye (2004), we know that for any $A \in \mathcal{D}(n; q^s)$ without replicates,

$$\sum_{r=1}^{s} \alpha_r(A) = \frac{q^s}{n} - 1.$$

So for any r, $n^2\alpha_r(A) \leq n(q^s - n)$. Thus from (12) and Lemma 3, the conclusion can be reached easily.

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